

# AFFINE TORIC $SL(2)$ - EMBEDDINGS

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**ABSTRACT.** In 1973 V.L.Popov classified affine  $SL(2)$ -embeddings. He proved that a locally transitive  $SL(2)$ -action on a normal affine three-dimensional variety  $X$  is uniquely determined by a pair  $(\frac{p}{q}, r)$ , where  $0 < \frac{p}{q} \leq 1$  is an uncanceled fraction and  $r$  is a positive integer. Here  $r$  is the order of the stabilizer of a generic point. In this paper we show that the variety  $X$  is toric, i.e. admits a locally transitive action of an algebraic torus, if and only if  $r$  is divisible by  $q - p$ . To do this we prove the following necessary and sufficient condition for an affine  $G/H$ -embedding to be toric. Suppose  $X$  is a normal affine variety,  $G$  is a simply connected semisimple algebraic group acting regularly on  $X$ ,  $H$  is a closed subgroup of  $G$  such that the character group  $\mathfrak{X}(H)$  is finite and  $G/H \hookrightarrow X$  is a dense open equivariant embedding. Then  $X$  is toric if and only if there exist a quasitorus  $\hat{T}$  and a  $(G \times \hat{T})$ -module  $V$  such that  $X \cong^G V // \hat{T}$ . The key role in the proof plays D. Cox's construction.

## INTRODUCTION

We work over an algebraically closed field  $\mathbb{K}$  of characteristic zero.

A normal variety  $X$  is called *toric* if it admits a locally transitive action of an algebraic torus  $T$ . Let us fix the action of  $T$  on  $X$ . The theory of toric varieties (see, for example, [1]) assigns a rational strongly convex polyhedral cone to an affine toric variety. This cone uniquely determines the variety up to a  $T$ -equivariant isomorphism.

It follows from D. Cox's paper [2] that an affine toric variety without non-constant invertible functions can be realized as a categorical quotient  $V // \hat{T}$  of a vector space  $V$  by a linear action of a quasitorus  $\hat{T}$ . This vector space is the spectrum of the *Cox ring*. On the other hand, it is not difficult to show that a categorical quotient of a vector space by a linear action of a quasitorus is toric.

Further all varieties are normal, affine and irreducible, and all actions are regular.

Suppose  $X$  is a variety,  $G$  is a simply connected semisimple group acting on  $X$ , and  $H$  is a closed subgroup of  $G$  such that the character group  $\mathfrak{X}(H)$  is finite. Let  $G/H \hookrightarrow X$  be an open equivariant embedding. The main idea of this paper consists in the following necessary and sufficient condition for  $X$  to be toric.

**Proposition.** *Affine  $G$ -variety  $X$  with an open  $G$ -equivariant embedding  $G/H \hookrightarrow X$  is toric if and only if there exist a quasitorus  $\hat{T}$  and a  $(G \times \hat{T})$ -module  $V$  such that  $X \cong^G V // \hat{T}$ .*

The proof is based on the construction of  $(G \times \hat{T})$ -action on the spectrum of the Cox ring. It follows from [3] that this action is linear.

In Section 6 we investigate three-dimensional varieties with a locally transitive  $SL(2)$ -action. Recall that a (normal affine irreducible) three-dimensional variety

$X$  with locally transitive and locally free  $\mathrm{SL}(2)$ -action is called an *affine*  $\mathrm{SL}(2)$ -embedding. Affine  $\mathrm{SL}(2)$ -embeddings are studied in [4] (see also [5, Ch.III, § 4]). In [4] it is also proved that if an  $\mathrm{SL}(2)$ -action on a three-dimensional variety  $X$  is locally transitive but not transitive, then the stabilizer of a generic point is a cyclic group. By  $r$  we denote the order of this group. Then the variety  $X$  is called an *affine*  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding. Let  $U$  be a maximal unipotent subgroup of  $\mathrm{SL}(2)$ . The algebra of  $U$ -invariants of the action is a monomial subalgebra of a polynomial algebra in two variables. This subalgebra determines a rational number  $0 < \frac{p}{q} \leq 1$ , which is called the *height* of the action. We shall assume that  $p$  and  $q$  are relatively prime. One of the main results of V.L.Popov's theory asserts that affine  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embeddings considered up to an  $\mathrm{SL}(2)$ -equivariant isomorphism are in one-to-one correspondence with pairs  $(\frac{p}{q}, r)$ ,  $0 < \frac{p}{q} \leq 1$ ,  $r \in \mathbb{N}$ . In this paper we find all pairs  $(\frac{p}{q}, r)$  such that the corresponding variety  $X$  is toric.

In Section 6 we use the previous proposition for  $G = \mathrm{SL}(2)$ ,  $H = \mathbb{Z}_r$ . Further we find out when the  $\mathrm{SL}(2)$ -action on  $V//\widehat{T}$  is locally transitive and determine the corresponding pairs  $(\frac{p}{q}, r)$ . Finally we obtain the following theorem.

**Theorem.** *Suppose  $X$  is a three-dimensional normal affine irreducible variety with a regular locally transitive  $\mathrm{SL}(2)$ -action. Then  $X$  is toric if and only if it is an  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding with height  $\frac{p}{q}$ ,  $(p, q) = 1$ , such that  $r$  is divisible by  $q - p$ .*

In Section 7 the cone of the  $\mathrm{SL}(2)/\mathbb{Z}_{j(q-p)}$ -embedding regarded as a toric variety is calculated.

After this paper had already been written, D.I.Panyushev informed the author that the easy part of this theorem claiming that  $\mathrm{SL}(2)$ -embeddings with  $r$  divisible by  $q - p$  are toric can be deduced from his results [6].

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## N o t a t i o n

$\mathrm{SL}(2)$  is the group of matrices with determinant 1 over the field  $\mathbb{K}$ ;

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, a \in \mathbb{K} \right\};$$

$\mathfrak{X}(G)$  is the group of characters of a group  $G$ ;

$\mathbb{Z}_r$  is the cyclic group of order  $r$ ;

$A^\times$  is the set of invertible elements of an algebra  $A$ ;

$\mathrm{Spec}(A)$  is the spectrum of a finitely generated algebra  $A$ ;

$\mathbb{K}[X]$  is the algebra of regular functions on a variety  $X$ ;

$\mathbb{K}(X)$  is the field of rational functions on a variety  $X$ ;

$(f)$  is a principal divisor of a function  $f \in \mathbb{K}(X)$ ;

$X//G$  is the categorical quotient of a variety  $X$  by an action of a group  $G$ ;

$Cl(X)$  is the divisor class group of a variety  $X$ .

## 1. PRELIMINARY INFORMATION ON TORIC VARIETIES

Let  $X$  be an (affine) toric variety with a torus  $T$  acting on it. By  $M$  denote the lattice  $\mathfrak{X}(T)$ . Let  $N = \mathrm{Hom}(M, \mathbb{Z})$  be the dual lattice, with the pairing between  $M$  and  $N$  denoted by  $\langle \cdot, \cdot \rangle$ . Then  $N$  may be identified with the lattice of one-parameter subgroups of  $T$ . Each  $m \in M$  gives a character  $\chi^m: T \rightarrow \mathbb{K}^*$ . The variety  $X$

corresponds to a strongly convex polyhedral cone  $\sigma$  in  $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ . Suppose  $\hat{\sigma}$  is the dual cone in  $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then  $\mathbb{K}[X]$  is isomorphic to the semigroup algebra  $\mathbb{K}[M \cap \hat{\sigma}]$ .

Let  $\Delta(1)$  be the set of one-dimensional faces of  $\sigma$ . It follows from [1, Ch.3, § 3] that each one-dimensional face  $\rho \in \Delta(1)$  corresponds to a prime  $T$ -invariant divisor  $D_{\rho}$  in  $X$  and each prime  $T$ -invariant divisor in  $X$  coincides with  $D_{\rho}$  for some  $\rho \in \Delta(1)$ .

The following lemma can be found in [1, Ch.3, § 4].

**Lemma 1.** *Any divisor on  $X$  is linearly equivalent to a  $T$ -invariant divisor.*

Let  $n_{\rho}$  be a unique generator of  $\rho \cap N$ , where  $\rho \in \Delta(1)$ . Take  $m \in M$ . Then  $\chi^m : T \rightarrow \mathbb{K}^*$  is a regular function on  $T$  and a rational function on  $X$ . It is proved in [1, Ch.3, § 3] that  $(\chi^m) = \sum \langle m, n_{\rho} \rangle D_{\rho}$ .

The following construction is given in [2].

**Definition 1.** The Cox ring of a toric variety  $X$  is a polynomial ring in  $u$  variables enumerated by elements of  $\Delta(1)$ , where  $u$  is the number of one-dimensional faces in  $\sigma$ :

$$\text{Cox}(X) = \mathbb{K}[x_{\rho} \mid \rho \in \Delta(1)].$$

A monomial  $\prod x_{\rho}^{a_{\rho}}$  corresponds to the  $T$ -invariant divisor  $D = \sum a_{\rho} D_{\rho}$ . Let  $x^D$  be this monomial. Let us grade  $\text{Cox}(X)$  by the group  $\text{Cl}(X)$  so that the degree of a monomial  $x^D$  is  $[D] \in \text{Cl}(X)$ . Degrees of two monomials are equal if these monomials are  $\prod x_{\rho}^{a_{\rho}}$  and  $\prod x_{\rho}^{a_{\rho} + \langle m, n_{\rho} \rangle}$  for some  $m \in M$ .

## 2. THE TOTAL COORDINATE RING

Given an arbitrary variety  $X$  and a divisor  $D$  on it we denote by  $H^0(X, D)$  the following subspace in  $\mathbb{K}(X)$ :

$$H^0(X, D) = \{f \in \mathbb{K}(X) \mid (f) + D \geq 0\}.$$

If  $X$  is toric and  $D = \sum a_{\rho} D_{\rho}$  is a  $T$ -invariant divisor, we have

$$H^0(X, D) = \bigoplus_{m \in P_D} \mathbb{K} \chi^m,$$

where  $P_D = \{m \in M \mid \forall \rho \in \Delta(1) : \langle m, n_{\rho} \rangle \geq -a_{\rho}\}$ .

Let  $X$  be an arbitrary variety. Suppose that  $\text{Cl}(X)$  is a free finitely generated group. Let  $\{\alpha_1, \dots, \alpha_s\}$  be its basis and  $L_{\alpha_i}$  be fixed divisors such that  $[L_{\alpha_i}] = \alpha_i$ . If  $\alpha \in \text{Cl}(X)$ , then we have  $\alpha = \sum l_i \alpha_i$ ,  $l_i \in \mathbb{Z}$ . We define  $L_{\alpha}$  as  $\sum l_i L_{\alpha_i}$ . Then we obtain a vector space graded by the group  $\text{Cl}(X)$ :

$$S(X) = \bigoplus_{\alpha \in \text{Cl}(X)} H^0(X, L_{\alpha}).$$

Let us define a multiplication in  $S(X)$  as follows. If  $f \in H^0(X, L_{\alpha})$ ,  $g \in H^0(X, L_{\beta})$ , then their product in  $S(X)$  is an element of  $H^0(X, L_{\alpha+\beta})$  equal to their product in  $\mathbb{K}(X)$ . This multiplication is extended to  $S(X)$  by distributivity. The ring  $S(X)$  is called *the total coordinate ring* of the variety  $X$ . It does not depend on basis chosen in  $\text{Cl}(X)$  and on the choice of divisors  $L_{\alpha_i}$  (see [7, 8]). If  $X$  is a toric variety we can choose  $L_{\alpha_i}$  to be  $T$ -invariant.

For the following lemma, see [2].

**Lemma 2.** *Assume that  $\mathbb{K}[X]^\times = \mathbb{K}^\times$  for a toric variety  $X$ . Then the graded algebras  $S(X)$  and  $\text{Cox}(X)$  are isomorphic.*

The next lemma is a simple corollary of the Rosenlicht theorem (see [9]).

**Lemma 3.** *Assume that a semisimple group  $G$  acts locally transitively on a variety  $X$ . Then there are no invertible non-constant functions in  $\mathbb{K}[X]$ .*

**Corollary 1.** *If for a toric variety  $X$  the conditions of Lemma 3 are fulfilled, then the algebra  $S(X)$  is free.*

Next we consider the problem of lifting of an action of an algebraic group  $G$  on a variety  $X$  to an action on the spectrum of the ring  $S(X)$ . In this context we can partially invert Corollary 1.

**Proposition 1.** *Let  $G$  be a simply connected semisimple group acting on a variety  $X$  and  $H \subset G$  be a closed subgroup. Assume that the character group  $\mathfrak{X}(H)$  is trivial. Suppose  $G/H \hookrightarrow X$  is an open equivariant embedding. Then  $\text{Cl}(X)$  is free finitely generated and the following conditions are equivalent:*

1) *the variety  $X$  is toric;*

2) *there exist a  $G$ -module  $V$  and a linear action of a torus  $\widehat{T} : V \xrightarrow{G}$  commuting with the action of  $G$  such, that  $X \cong V // \widehat{T}$ ;*

3) *the algebra  $S(X)$  is free.*

*Proof.* First we prove that  $\text{Cl}(X)$  is free finitely generated. Denote by  $E_1, E_2 \dots E_s$  all prime divisors in  $X \setminus (G/H)$ . Since  $G$  is connected, all the divisors  $E_i$  are  $G$ -invariant. Since  $G$  is a simply connected semisimple group and  $H$  is its closed subgroup,  $\text{Cl}(G/H) \cong \text{Pic}(G/H) \cong \mathfrak{X}(H)$  (see [10, Prop. 1], [10, Th. 4]). Therefore,  $\text{Cl}(G/H) = 0$ . Hence  $\text{Cl}(X)$  is generated by  $G$ -invariant divisors, that is, by divisors  $\sum a_i E_i$ . If  $c_1 E_1 + \dots + c_s E_s = (f)$ , then, since  $E_i$  are  $G$ -invariant,  $f$  is  $G$ -semiinvariant. But, since  $G$  is semisimple, we have  $\mathfrak{X}(G) = 0$  and hence  $f$  is  $G$ -invariant. This implies  $f = \text{const}$ , and  $\text{Cl}(X)$  is freely generated by the divisor classes  $[E_i]$ .

Implication 1) $\Rightarrow$  3) is already proved in Corollary 1.

3) $\Rightarrow$  2) The group  $G$  acts on  $\mathbb{K}(X)$  as  $g \cdot f(x) = f(g^{-1} \cdot x)$ . Consider  $D = \sum a_i E_i$  and  $f \in H^0(X, D)$ . Then

$$(f) + D \geq 0;$$

$$(g \cdot f) + D \geq 0 \Rightarrow g \cdot f \in H^0(X, D),$$

that is,  $H^0(X, D)$  is a  $G$ -module. Obviously, the  $G$ -action respects the multiplication in  $S(X)$ .

Let us choose  $L_{[E_i]} = E_i$ . Then

$$S(X) = \bigoplus_{\alpha \in \text{Cl}(X)} H^0(X, L_\alpha) = \bigoplus_{D = a_1 E_1 + \dots + a_s E_s} H^0(X, D).$$

Consider an  $s$ -dimensional torus  $\widehat{T}$ . Its action on  $H^0(X, D)$  is defined as follows. If  $t = (t_1, \dots, t_s) \in \widehat{T}$ , then  $t \cdot f = t_1^{a_1} \dots t_s^{a_s} f$ , while  $f \in H^0(X, D)$ ,  $D = a_1 E_1 + \dots + a_s E_s$ . This action is extended to  $S(X)$  by linearity. The actions of  $G$  and  $\widehat{T}$  on  $S(X)$  commute. Therefore,

$$(\widehat{T} \times G) : \text{Spec } S(X).$$

The only  $G \times \widehat{T}$ -invariant functions on  $S(X)$  are constants. Indeed,

$$S(X)^{\widehat{T} \times G} = (S(X)^{\widehat{T}})^G$$

and

$$S(X)^{\widehat{T}} = H^0(X, 0) \cong \mathbb{K}[X].$$

The  $G$ -action on  $X$  has an open orbit. Hence,  $\mathbb{K}[X]^G = \mathbb{K}$ . Since  $S(X)$  is free, we have  $\text{Spec } S(X) \cong \mathbb{K}^u$ . By Kraft-Popov's theorem [3], an action of a reductive group on  $\mathbb{K}^u$  is equivalent to a linear action, if the regular functions invariant under this action are only constants. Now denote the  $(\widehat{T} \times G)$ -module  $\text{Spec } S(X)$  by  $V$ .

Since  $S(X)^{\widehat{T}} = \mathbb{K}[X]$ ,  $X \cong V // \widehat{T}$ .

2) $\Rightarrow$  1) Since the  $\widehat{T}$ -action on  $V$  is linear, all elements of  $\widehat{T}$  are simultaneously diagonalizable. Consider the torus  $\overline{T}$  consisting of all diagonal operators in the same basis in which  $\widehat{T}$  is diagonal. Then the action  $T = \overline{T} / \widehat{T} : V // \widehat{T}$  has an open orbit. Hence  $X$  is a toric variety.  $\square$

### 3. LIFTING OF THE ACTION

It follows from the proof of Corollary 1 that, if  $\mathfrak{X}(H) = 0$ , then the action  $G : X$  may be lifted to an action on the spectrum of the total coordinate ring. In this section we define a construction similar to the construction of total coordinate ring and consider the problem of lifting of the action, when the group of characters  $\mathfrak{X}(H)$  is finite.

Suppose that the divisor class group of a variety  $X$  is finitely generated. Let us choose a generative system  $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_s$  of the group

$$\text{Cl}(X) \cong \mathbb{Z}^n \oplus \mathbb{Z}_{k_1} \oplus \dots \oplus \mathbb{Z}_{k_s},$$

where  $\{\xi_i\}$  is a basis of the free group  $\mathbb{Z}^n$ , and  $\eta_j$  is a generator of the cyclic group  $\mathbb{Z}_{k_j}$ . Let us fix some divisors  $E_1, \dots, E_n, W_1, \dots, W_s$  on  $X$ , such that  $[E_i] = \xi_i$ ,  $[W_j] = \eta_j$ . Now we can define a generalisation of total coordinate ring. Consider a linear space

$$S(X) = \bigoplus_{\lambda_i \in \mathbb{Z}, \mu_j = 0, 1, \dots, k_j - 1} H^0(X, \sum \lambda_i E_i + \sum \mu_j W_j).$$

Let us choose rational functions  $F_1, \dots, F_s$  such that  $k_s W_s = (F_s)$ . We define a multiplication  $*$  on  $S(X)$  as follows. If  $f \in H^0(X, \sum a_i E_i + \sum b_j W_j)$  and  $g \in H^0(X, \sum c_i E_i + \sum d_j W_j)$ , then

$$f * g = fg \prod F_j^{\left[\frac{b_j + d_j}{k_j}\right]} \in H^0\left(X, \sum (a_i + c_i) E_i + \sum \left\{\frac{b_j + d_j}{k_j}\right\} k_j W_j\right),$$

where  $[x]$  is the integer part of  $x$  and  $\{x\}$  is the fractional part of  $x$ . We need to prove that  $fg \prod F_j^{\left[\frac{b_j + d_j}{k_j}\right]}$  belongs to

$$H^0\left(X, \sum (a_i + c_i) E_i + \sum \left\{\frac{b_j + d_j}{k_j}\right\} k_j W_j\right).$$

It follows from

$$\begin{aligned} (f) + \sum a_i E_i + \sum b_j W_j &\geq 0; \\ (g) + \sum c_i E_i + \sum d_j W_j &\geq 0; \end{aligned}$$

$$\begin{aligned}
& (fg \prod F_j^{\lfloor \frac{b_j+d_j}{k_j} \rfloor}) + \sum (a_i + c_i)E_i + \sum \left\{ \frac{b_j + d_j}{k_j} \right\} k_j W_j = \\
& = (fg) + \sum (a_i + c_i)E_i + \sum (b_j + d_j)W_j \geq 0.
\end{aligned}$$

Extending  $*$  by distributivity, we obtain a commutative algebra  $S(X)$ . The algebra  $S(X)$  is graded by the group  $\text{Cl}(X)$ .

**Lemma 4.** *The algebra  $S(X)$  does not depend on the choice of  $E_1, \dots, E_n, W_1, \dots, W_s$ .*

*Proof.* Let  $\tilde{E}_1, \dots, \tilde{E}_n, \tilde{W}_1, \dots, \tilde{W}_s$  be other divisors on  $X$  such that  $[\tilde{E}_i] = \xi_i$ ,  $[\tilde{W}_j] = \eta_j$ . Then  $\tilde{E}_i = E_i - (J_i)$ ,  $\tilde{W}_j = W_j - (R_j)$ . There exists a map

$$\varphi: H^0(X, \sum \lambda_i E_i + \sum \mu_j W_j) \rightarrow H^0(X, \sum \lambda_i \tilde{E}_i + \sum \mu_j \tilde{W}_j); f \mapsto f \prod J_i^{\lambda_i} \prod R_j^{\mu_j}.$$

Let us combine the maps  $\varphi$  with different  $\lambda_i$  and  $\mu_j$  and extend this map to a linear map. We obtain a map between the algebras  $S(X)$  corresponding to different choices of divisors  $E_1, \dots, E_n, W_1, \dots, W_s$ . Suppose  $f \in H^0(X, \sum a_i E_i + \sum b_j W_j)$ ,  $h \in H^0(X, \sum c_i E_i + \sum d_j W_j)$ . Then

$$\begin{aligned}
\varphi(f) * \varphi(g) &= \varphi(f)\varphi(g) \prod (F_j R_j^{-k_j})^{\lfloor \frac{b_j+d_j}{k_j} \rfloor} = \\
&= \left( f \prod J_i^{a_i} \prod R_j^{b_j} \right) \left( h \prod J_i^{c_i} \prod R_j^{d_j} \right) \prod (F_j R_j^{-k_j})^{\lfloor \frac{b_j+d_j}{k_j} \rfloor} = \\
&= \left( fh \prod F_j^{\lfloor \frac{b_j+d_j}{k_j} \rfloor} \right) \prod J_i^{a_i+c_i} \prod R_j^{\lfloor \frac{b_j+d_j}{k_j} \rfloor k_j} = \varphi(f * h).
\end{aligned}$$

Hence,  $\varphi$  is an isomorphism.  $\square$

*Remark 1.* The algebra  $S(X)$  does not depend on a splitting of  $\text{Cl}(X)$  into a direct sum of cyclic groups and on the choice of a system of generators of the group  $\text{Cl}(X)$ .

**Proposition 2.** *Let  $G$  be a semisimple simply connected group acting on a variety  $X$  and  $H \subset G$  be a closed subgroup. Assume that the group of characters  $\mathfrak{X}(H)$  is finite. Suppose  $G/H \hookrightarrow X$  is an open equivariant embedding. Then  $\text{Cl}(X)$  is finitely generated and there exists an action  $G : S(X)$  preserving homogeneous components and coinciding on  $S(X)_0 \cong \mathbb{K}[X]$  with the  $G$ -action on  $\mathbb{K}[X]$ .*

*Proof.* We denote by  $E_1, \dots, E_n$  all prime divisors in  $X \setminus (G/H)$ . Since  $G$  is semisimple, these divisors are  $G$ -invariant. From [10, Prop. 1] and [10, Th. 4] it follows that  $\text{Cl}(G/H) \cong \mathfrak{X}(H)$ . The divisors  $E_1, \dots, E_n$  generate a free subgroup  $\mathbb{Z}^n \subset \text{Cl}(X)$ . For any divisor  $D$  in  $X$  there exists a linear combination  $\sum \lambda_i E_i$  such that the support of  $D - \sum \lambda_i E_i$  lies in  $G/H$ . Hence, the quotient group  $\text{Cl}(X)/\mathbb{Z}^n$  is finite. Therefore, we have

$$\text{Cl}(X) = \mathbb{Z}^n \oplus \mathbb{Z}_{k_1} \oplus \dots \oplus \mathbb{Z}_{k_s}.$$

For any  $\alpha \in \text{Cl}(X)$  we have  $\alpha = (\beta_1, \dots, \beta_n, \alpha_1, \dots, \alpha_s)$ , where  $\beta_i \in \mathbb{Z}$ ,  $\alpha_j \in \mathbb{Z}_{k_j}$ . Let us consider divisors  $W_1, \dots, W_s$  such that  $[W_j] = (0, \dots, 0, 1, 0, \dots, 0)$ , where 1 stands at the  $(n+j)$ th position.

By [10, Prop. 1],  $\text{Cl}(G) = 0$ . This means that all divisors in  $G$  are principal. Suppose  $\pi: G \rightarrow G/H$  is the factorisation morphism and  $i: G/H \rightarrow X$  is the embedding. These morphisms induce the embeddings of fields  $\mathbb{K}(X) \xrightarrow{i^*} \mathbb{K}(G/H) \xrightarrow{\pi^*}$

$\mathbb{K}(G)$ . The homogeneous space  $G/H$  is smooth. Hence any Weil divisor in  $G/H$  is a Cartier divisor. Since  $W_j \cap (G/H)$  is a Cartier divisor, there exists an open covering  $\{U_\alpha\}$  of  $G/H$  such that  $W_j \cap U_\alpha = (\varphi_\alpha)$ . Let  $Q$  be the pullback of  $W_j \cap (G/H)$  in  $G$ . We have  $Q \cap \pi^{-1}(U_\alpha) = (\pi^*(\varphi_\alpha))$ . Since  $Q$  is a principal divisor, we have  $Q = (f_j)$ , where  $f_j \in \mathbb{K}[G]$ . Therefore the pullback of the divisor  $(F_j|_{G/H}) = (i^*(F_j))$  is  $(f_j^{k_j})$ , that is,  $(\pi^*(i^*(F_j))) = (f_j^{k_j})$ . We may assume that  $\pi^*(i^*(F_j)) = f_j^{k_j}$ . In the sequel we shall not distinguish between functions of  $\mathbb{K}(X)$  and their images under the map  $\pi^* \circ i^*$ . Thus,  $f_j$  is an element of  $\mathbb{K}(G)$  such that  $f_j^{k_j} \in \mathbb{K}(X)$  and  $k_j W_j = (f_j^{k_j})$ .

Let us define an action  $\bullet$  of the group  $G$  on  $H^0(X, D)$ , where  $D = \sum \lambda_i E_i + \sum \mu_j W_j$ ,  $\lambda_i \in \mathbb{Z}$ ,  $\mu_j = 0, \dots, k_j - 1$ . Suppose  $g \in G$ . Then  $g^{-1} \cdot D = \sum \lambda_i E_i + \sum \mu_j (g^{-1} \cdot W_j)$ . Since  $k_j W_j = (f_j^{k_j})$ , we obtain  $k_j (g^{-1} \cdot W_j) = g^{-1} \cdot (k_j W_j) = g^{-1} \cdot (f_j^{k_j}) = (g^{-1} \cdot f_j)^{k_j} = (f_j^{k_j}) + \left( \frac{g^{-1} \cdot f_j}{f_j} \right)^{k_j}$ . Let us consider the  $H$ -action on  $G$  by right shifts. Then  $H$  acts on  $\mathbb{K}(G)$  and  $\mathbb{K}(G/H) = \mathbb{K}(G)^H$ . Suppose  $h \in H$ . Since  $f_j^{k_j} \in \mathbb{K}(G/H)$ , we have  $h \cdot f_j = \varepsilon f_j$ , where  $\varepsilon^{k_j} = 1$ . Then

$$h \cdot \frac{g^{-1} \cdot f_j}{f_j}(g') = \frac{f_j(h^{-1} g g')}{f_j(h^{-1} g')} = \frac{\varepsilon f_j(g g')}{\varepsilon f_j(g')} = \frac{g^{-1} \cdot f_j}{f_j}(g').$$

Hence,  $\frac{g^{-1} \cdot f_j}{f_j} \in \mathbb{K}(G)^H = \mathbb{K}(X)$ . By definition, we put

$$g \bullet f = (g \cdot f) \prod \left( \frac{g \cdot f_j}{f_j} \right)^{\mu_j} = g \cdot \left( \prod \left( \frac{f_j}{g^{-1} \cdot f_j} \right)^{\mu_j} f \right),$$

where  $f \in H^0(X, D)$ , and the action  $\cdot$  is the standard action on  $\mathbb{K}(G)$ . Let us check that  $g \bullet f$  lies in  $H^0(X, D)$ . We have  $(f) + D \geq 0$ . Hence,  $\left( f \prod \left( \frac{f_j}{g^{-1} \cdot f_j} \right)^{\mu_j} \right) + g^{-1} \cdot D \geq 0$ . Therefore,  $g \cdot \left( f \prod \left( \frac{f_j}{g^{-1} \cdot f_j} \right)^{\mu_j} \right) + D \geq 0$ , that is,  $g \bullet f \in H^0(X, D)$ . Let us check that  $\bullet$  is an action.

$$\begin{aligned} (g\tilde{g}) \bullet f &= (g\tilde{g}) \left( f \prod \left( \frac{f_j}{(g\tilde{g})^{-1} \cdot f_j} \right)^{\mu_j} \right) = \\ &= g \cdot \left( \tilde{g} \cdot \left( f \prod \left( \frac{f_j}{\tilde{g}^{-1} \cdot f_j} \right)^{\mu_j} \prod \left( \frac{\tilde{g}^{-1} \cdot f_j}{\tilde{g}^{-1} \cdot (g^{-1} \cdot f_j)} \right)^{\mu_j} \right) \right) = \\ &= g \cdot \left( (\tilde{g} \bullet f) \left( \frac{f_j}{g^{-1} \cdot f_j} \right)^{\mu_j} \right) = g \bullet (\tilde{g} \bullet f). \end{aligned}$$

For  $(\mu_1, \dots, \mu_s) = (0, \dots, 0)$  the action  $\bullet$  coincides with the action  $\cdot$  on  $\mathbb{K}[X]$ . Let us extend the action  $\bullet$  to the linear action on  $S(X)$ .

Let us prove that  $\bullet$  is an action by automorphisms of algebra  $S(X)$ . We need to show that  $g \bullet (f * f') = (g \bullet f) * (g \bullet f')$  for any  $g \in G$  and for any  $f$  and  $f'$  in  $S(X)$ . Since the action  $\bullet$  is linear we may check only  $g \bullet (\chi^m * \chi^n) = (g \bullet \chi^m) * (g \bullet \chi^n)$ , where  $\chi^m \in H^0(X, \sum a_i E_i + \sum b_j W_j)$ ,  $\chi^n \in H^0(X, \sum c_i E_i + \sum d_j W_j)$ . By definition,

$$\chi^m * \chi^n = \chi^m \chi^n \prod F_j^{\lfloor \frac{b_j + d_j}{k_j} \rfloor} \in H^0 \left( X, \sum (a_i + c_i) E_i + \left\{ \frac{b_j + d_j}{k_j} \right\} k_j W_j \right).$$

We have

$$\begin{aligned}
g \bullet (\chi^m * \chi^n) &= \left( g \cdot \left( \chi^m \chi^n \prod_{F_j}^{[\frac{b_j+d_j}{k_j}]} \right) \right) \prod \left( \frac{g \cdot f_i}{f_i} \right)^{\{\frac{b_j+d_j}{k_j}\}k_j} = \\
&= \frac{(g \cdot (\chi^m \chi^n)) \prod (g \cdot f_j)^{b_j+d_j}}{\prod f_j^{\{\frac{b_j+d_j}{k_j}\}k_j}} = \\
&= \left( (g \cdot \chi^m) \prod \left( \frac{g \cdot f_i}{f_i} \right)^{b_j} \right) \left( (g \cdot \chi^n) \prod \left( \frac{g \cdot f_i}{f_i} \right)^{d_j} \right) \prod_{F_j}^{[\frac{b_j+d_j}{k_j}]} = (g \bullet f) * (g \bullet f').
\end{aligned}$$

Proposition 2 is proved.  $\square$

If the algebra  $S(X)$  is finitely generated we can consider  $\text{Spec } S(X)$ .

**Lemma 5.** *The action  $\bullet$  on  $S(X)$  induces a regular action on  $\text{Spec } S(X)$ .*

*Proof.* We need to prove that  $S(X)$  is a rational  $G$ -module. Let us prove that  $H^0(X, D)$  is a rational  $G$ -module with respect to the action  $\bullet$ . Here  $D = \sum a_i E_i + \sum b_j W_j$  and the divisors  $E_i$  are  $G$ -invariant. Let us recall that there are embeddings of fields  $\mathbb{K}(X) \xrightarrow{i^*} \mathbb{K}(G/H) \xrightarrow{\pi^*} \mathbb{K}(G)$ . If  $f \in H^0(X, D)$ , then  $i^*(f) \in H^0(G/H, D \cap (G/H))$ . The pullback of the divisor  $D \cap (G/H)$  under the map  $\pi$  is a principal divisor  $(J)$  in  $G$ . Hence we obtain  $f \in H^0(G, (J))$ , that is,  $fJ \in \mathbb{K}[G]$ . (Recall that we do not distinguish between  $f \in \mathbb{K}(X)$  and  $\pi^*(i^*(f)) \in \mathbb{K}(G)$ .) The action  $\bullet$  on  $H^0(X, D)$  is defined by the formula

$$g \bullet f = g \cdot \left( f \prod \left( \frac{f_j}{g^{-1} f_j} \right)^{b_j} \right) = \frac{g \cdot (f \prod f_j^{b_j})}{\prod f_j^{b_j}}.$$

By definition,  $\pi^*(W_j \cap (G/H)) = (f_j)$ . Hence,  $\pi^*(D \cap (G/H)) = (\prod f_j^{b_j})$ , that is, we may assume  $J = \prod f_j^{b_j}$ . Then  $g \bullet f = \frac{g \cdot (fJ)}{J}$ . But it has been proved above that  $fJ \in \mathbb{K}[G]$ . It is known that  $\mathbb{K}[G]$  is a rational module. Therefore,  $H^0(X, D)$  is a rational  $G$ -module with respect to the action  $\bullet$ . Hence,  $S(X)$  is a rational  $G$ -module.  $\square$

Let  $X$  be a toric variety. Then Lemmas 1 and 4 imply that the divisors  $E_1, \dots, E_n, W_1, \dots, W_s$  in the definition of  $S(X)$  can be chosen to be  $T$ -invariant.

**Lemma 6.** *The graded algebras  $S(X)$  and  $\text{Cox}(X)$  are isomorphic.*

*Proof.* Consider a divisor  $D = \sum a_i E_i + \sum b_j W_j$ . Recall that

$$H^0(X, D) = \bigoplus_{m \in P_D} \mathbb{K} \chi^m.$$

Let  $\psi_D$  be the linear map from  $H^0(X, D)$  to  $\text{Cox}(X)_{[D]}$  taking each  $\chi^m$ , where  $m \in P_D$ , to  $x^{D+(\chi^m)}$ . Since the only invertible functions in  $\mathbb{K}[X]$  are constants, different functions  $\chi^m$  corresponds to different divisors  $(\chi^m)$ . Therefore,  $\psi_D$  is an embedding. If  $x^{D_1} \in \text{Cox}(X)_{[D]}$ , then  $D_1 - D$  is a principal divisor. Suppose  $D = \sum \alpha_\rho D_\rho$ ,  $D_1 = \sum \beta_\rho D_\rho$ . Then  $\beta_\rho - \alpha_\rho = \langle m_0, n_\rho \rangle$  for some  $m_0 \in M$ . But  $\prod x_\rho^{\beta_\rho}$  is a monomial in  $\text{Cox}(X)$ . Hence for each  $\rho \in \Delta(1)$  we have  $\beta_\rho \geq 0$ . Therefore  $\langle m_0, n_\rho \rangle \geq -\alpha_\rho$ , that is,  $m_0$  lies in  $P_D$ . Consequently  $\psi_D$  is a surjection. Thus  $\psi_D$  is an isomorphism of linear spaces. Let us combine all maps  $\psi_{(\sum a_i E_i + \sum b_j W_j)}$ ,  $a_i \in$



$\mathbb{Z}, b_j = 0, \dots, k_j - 1$ . We obtain the isomorphism of linear spaces  $\psi: S(X) \rightarrow \text{Cox}(X)$ . Let us prove that  $\psi$  is an isomorphism of algebras. We need to check that  $\psi(f * g) = \psi(f)\psi(g)$  for any  $f$  and  $g$  in  $S(X)$ . By distributivity we may assume that  $f = \chi^m \in H^0(X, \sum a_i E_i + \sum b_j W_j)$  and  $g = \chi^n \in H^0(X, \sum c_i E_i + \sum d_j W_j)$ . Then by the definitions of the multiplication  $*$  and of the isomorphism  $\psi$  we obtain

$$\begin{aligned} \psi(f * g) &= \psi(\chi^m * \chi^n) = \psi\left(\chi^{m+n} \prod F_j^{\lfloor \frac{b_j+d_j}{k_j} \rfloor}\right) = \\ &= x^{\sum (a_i+c_i)E_i + \sum \{\frac{b_j+d_j}{k_j}\}k_j W_j + (\chi^m) + (\chi^n) + \sum \lfloor \frac{b_j+d_j}{k_j} \rfloor (F_j)} = \\ &= x^{\sum (a_i+c_i)E_i + \sum \{\frac{b_j+d_j}{k_j}\}k_j W_j + (\chi^m) + (\chi^n) + \sum \lfloor \frac{b_j+d_j}{k_j} \rfloor k_j W_j} = \\ &= x^{\sum (a_i+c_i)E_i + \sum (b_j+d_j)W_j + (\chi^m) + (\chi^n)} = \\ &= x^{\sum a_i E_i + \sum b_j W_j + (\chi^m)} x^{\sum c_i E_i + \sum d_j W_j + (\chi^n)} = \psi(\chi^m)\psi(\chi^n) = \psi(f)\psi(g). \end{aligned}$$

□

#### 4. A NECESSARY AND SUFFICIENT CONDITION FOR AN AFFINE $G/H$ -EMBEDDING TO BE TORIC.

**Theorem 1.** *Let  $G$  be a semisimple simply connected group acting on a variety  $X$ . Suppose  $H$  is a closed subgroup in  $G$  such that the group of characters  $\mathfrak{X}(H)$  is finite. If there is an open equivariant embedding  $G/H \hookrightarrow X$ , then the following conditions are equivalent:*

- 1)  $X$  is a toric variety;
- 2) There exist a  $G$ -module  $V$  and a linear action of a quasitorus  $\widehat{T}$  on  $V$  commuting with the  $G$ -action such that  $X \cong V // \widehat{T}$ ;
- 3) The algebra  $S(X)$  is free.

*Proof.* The implication 1)  $\Rightarrow$  3) follows from Lemma 6.

3)  $\Rightarrow$  2) The algebra  $S(X)$  is graded by the group  $\text{Cl}(X) \cong \mathbb{Z}^n \oplus \mathbb{Z}_{k_1} \oplus \dots \oplus \mathbb{Z}_{k_s}$ . Proposition 2 implies that there exists a  $G$ -action by automorphisms of the ring  $S(X)$  that preserves homogeneous components and coincides on  $S(X)_{(0, \dots, 0)} \cong \mathbb{K}[X]$  with the standard  $G$ -action. Consider a quasitorus  $\widehat{T} = \widetilde{T} \times \mathbb{Z}_{k_1} \times \dots \times \mathbb{Z}_{k_s}$ , where  $\widetilde{T}$  is an  $n$ -dimensional torus. Let us define a  $\widehat{T}$ -action on  $S(X)$ . Suppose  $f \in S(X)_{(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_s)}$ , where  $\lambda_i \in \mathbb{Z}$ ,  $\mu_j = 0, 1, \dots, k_j - 1$ . Let  $t = (t_1, \dots, t_n) \in \widetilde{T}$ ,  $\varepsilon_j \in \mathbb{Z}_{k_j} \subset \mathbb{K}^*$ . Then  $(t, \varepsilon_1, \dots, \varepsilon_s) \cdot f = \prod t_i^{\lambda_i} \prod \varepsilon_j^{\mu_j} f$ . Obviously, this is an action. Moreover, this action commute with  $G$ -action, because  $G$  preserves homogeneous components in  $S(X)$  and every element of  $\widehat{T}$  acts on each component by multiplying by a constant. Hence  $G \times \widehat{T}$  acts on  $S(X)$ . Lemma 5 implies that  $G \times \widehat{T}$  acts on  $V = \text{Spec } S(X)$ . Since  $S(X)$  is a free algebra,  $V$  is a linear space. We have  $S(X)^{G \times \widehat{T}} = (S(X)^{\widehat{T}})^G = S(X)_{(0, \dots, 0)}^G = \mathbb{K}[X]^G = \mathbb{K}$ . The last equality holds because  $G$  acts on  $X$  with an open orbit. The Kraft-Popov theorem [3] implies that the action  $G \times \widehat{T}: V$  is equivalent to a linear action. Note that  $S(X)^{\widehat{T}} = S(X)_{(0, \dots, 0)}$ . Moreover, since  $G$ -action on  $S(X)_{(0, \dots, 0)} \cong \mathbb{K}[X]$  coincides with the standard action, we have  $S(X)^{\widehat{T}} = S(X)_{(0, \dots, 0)} \stackrel{G}{\cong} \mathbb{K}[X]$ . Hence  $V // \widehat{T} \stackrel{G}{\cong} X$ .

2)  $\Rightarrow$  1) The proof coincides with the proof of the corresponding implication in Proposition 1.  $\square$

*Remark 2.* It follows from the proofs of Proposition 2 and Theorem 1 that  $n$  is the number of prime divisors in the complement to  $G/H$  and  $s$  is not larger than the number of cyclic components in any splitting of  $\mathfrak{X}(H)$ .

## 5. NECESSARY INFORMATION ON $\mathrm{SL}(2)/\mathbb{Z}_r$ - EMBEDDINGS

The results mentioned in this section can be found in [4] (see also [5, chapter 3]).

A normal affine  $\mathrm{SL}(2)$ -embedding  $X$  is unique up to an isomorphism determined by its height. The height is a rational number defined as follows. Let us consider an  $\mathrm{SL}(2)$ -equivariant open embedding  $\varphi: \mathrm{SL}(2) \hookrightarrow X$ . It induces an embedding  $\varphi^*: \mathbb{K}[X] \hookrightarrow \mathbb{K}[\mathrm{SL}(2)] = \mathbb{K}[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1)$ . Here  $\alpha, \beta, \gamma$  and  $\delta$  are the functions such that  $\alpha(A) = a_{11}$ ,  $\beta(A) = a_{12}$ ,  $\gamma(A) = a_{21}$ ,  $\delta(A) = a_{22}$  for any  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2)$ . Let us consider the unipotent subgroup  $U = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$  in  $\mathrm{SL}(2)$ .  $U$  acts on  $\mathrm{SL}(2)$  by left shifts. The algebra of  $U$ -invariant functions on  $\mathrm{SL}(2)$  is  $\mathbb{K}[\mathrm{SL}(2)]^U = \mathbb{K}[\alpha, \beta]$ . Let us consider the restriction of  $\varphi^*$  to  $\mathbb{K}[X]^U$ . We have  $\varphi^*: \mathbb{K}[X]^U \hookrightarrow \mathbb{K}[\mathrm{SL}(2)]^U = \mathbb{K}[\alpha, \beta]$ .

**Proposition 3.** *The image  $\varphi^*(\mathbb{K}[X]^U)$  is a monomial subalgebra in  $\mathbb{K}[\alpha, \beta]$ . Moreover,  $\varphi^*(\mathbb{K}[X]^U) = \langle \alpha^i \beta^j \mid j/i \leq h \rangle$  for some rational number  $h$ .*

The rational number  $h$  is called the *height* of the embedding  $X$ .

For any  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding  $X$  there exists a unique up to an isomorphism  $\mathrm{SL}(2)$ -embedding  $Y$  such that  $X \cong Y//\mathbb{Z}_r$ . Here the  $\mathbb{Z}_r$ -action on  $Y$  is an extension of the  $\mathbb{Z}_r \subset \mathrm{SL}(2)$  on  $\mathrm{SL}(2)$  by right shifts. The height of the corresponding  $\mathrm{SL}(2)$ -embedding  $Y$  is called the *height* of the  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding  $X$ . The order of the stabilizer of a point in the open orbit is called the *degree* of  $X$ . Thus, any normal affine  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding is unique up to isomorphism determined by its height  $h \in \mathbb{Q} \cap (0, 1]$  and degree  $r \in \mathbb{N}$ .

**Proposition 4.** *There is a unique prime divisor in the complement of the open orbit in a normal affine  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding.*

The divisor class group of a normal affine  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding has been calculated in [6]. It is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_l$ , where  $l = \frac{r}{(r, q-p)}$ .

## 6. CLASSIFICATION OF THREE-DIMENSIONAL TORIC VARIETIES ADMITTING A LOCALLY TRANSITIVE $\mathrm{SL}(2)$ -ACTION

**Proposition 5.** *Let  $G$  be a semisimple simply connected group and  $H$  be a closed subgroup of  $G$ . Assume that  $\mathfrak{X}(H)$  is finite. If the homogeneous space  $G/H$  has positive dimension, then  $G/H$  is not a toric variety.*

*Proof.* Let  $X = G/H$  be a toric variety of positive dimension. It follows from Theorem 1 that  $X \cong \overset{G}{V//L}$ , where  $V$  is a vector space and  $L$  is a group acting linearly on  $V$ . Hence the image of  $0 \in V$  under the factorisation morphism is an  $G$ -stable point. This contradicts the transitivity of the  $G$ -action on  $X$ .  $\square$

**Corollary 2.** *No three-dimensional toric variety admits a transitive  $\mathrm{SL}(2)$ -action.*

Now suppose that the  $\mathrm{SL}(2)$ -action on a three-dimensional variety  $X$  is locally transitive but is not transitive. Then  $X$  is an  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding. Every  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding is uniquely determined by its height  $h$ . Let us calculate the heights of the toric  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embeddings. Let us consider a toric  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding  $X$ . A toric variety  $X$  corresponds to a cone  $\sigma$  in the space  $N_{\mathbb{Q}}$ .

**Lemma 7.** *The number  $u$  of one-dimensional edges in  $\sigma$  is equal to 4.*

*Proof.* By Lemma 1, any divisor in  $X$  is equivalent to a  $T$ -invariant one. Any  $T$ -invariant divisor in  $X$  can be written as  $\sum a_{\rho} D_{\rho}$ . Hence, the rank of the group of  $T$ -invariant Weil divisors in  $X$  is  $u$ . A  $T$ -invariant divisor is principal if and only if it can be written as  $\sum \langle m, n_{\rho} \rangle D_{\rho}$  for some  $m \in M$ . The lattice  $M$  has dimension three. Hence the rank of the group of principal  $T$ -invariant divisors is three. Therefore the rank of the divisor class group is  $u - 3 = 1$ , that is,  $u = 4$ .  $\square$

Recall that there exists a vector space  $V \cong \mathbb{K}^u = \mathbb{K}^4$  and a linear  $(\mathrm{SL}_2 \times \widehat{T})$ -action on  $V$  such that  $X \cong V//\widehat{T}$ . By Remark 2, we have  $\widehat{T} = \widetilde{T} \times \mathbb{Z}_l$ , where  $\widetilde{T}$  is a one-dimensional torus. Suppose  $V = V_1 \oplus \dots \oplus V_n$ , where  $V_i$  are the weight spaces of  $\widetilde{T}$ -action. Then each  $V_i$  is an  $\mathrm{SL}(2)$ -module.

Let us consider the space  $R_s = \langle x^s, x^{s-1}y, \dots, y^s \rangle$  of binary forms of degree  $s$ . The group  $\mathrm{SL}(2)$  acts on  $R_s$  by the following rule. If  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then  $g \cdot x = dx - by$ ;  $g \cdot y = -cx + ay$ . Then  $R_s$  is an irreducible  $\mathrm{SL}_2$ -module of dimension  $s + 1$ . Any irreducible  $(s + 1)$ -dimensional  $\mathrm{SL}_2$ -module is isomorphic to  $R_s$ .

There are 5 cases of splitting of  $V$  into irreducible  $\mathrm{SL}_2$ -modules.

- 1)  $V = R_0 \oplus R_0 \oplus R_0 \oplus R_0$ , 2)  $V = R_0 \oplus R_0 \oplus R_1$ , 3)  $V = R_0 \oplus R_2$ , 4)  $V = R_1 \oplus R_1$ , 5)  $V = R_3$ .

In cases 1), 2) and 3) all orbits in  $V$  have dimension less than three. Hence, all orbits in  $V//\widehat{T}$  have dimension less than three.

In case 5)  $\widetilde{T}$  acts on  $V$  by homotatis. Hence  $V//\widetilde{T}$  is a point. Thus case 5) is impossible.

Let us consider case 4):  $V = R_1 \oplus R_1$ . The variety  $V//\widetilde{T}$  can be three-dimensional only if the weights of the  $\widetilde{T}$ -action have opposite signs.

We have

$$\begin{aligned} V &= \mathbb{K}^2 \oplus \mathbb{K}^2 = V_1 \oplus V_2; \\ \widetilde{T} : V_1, \quad t \cdot v_1 &= t^{np} v_1, \quad p \in \mathbb{N}, \quad v_1 \in V_1; \\ \widetilde{T} : V_2, \quad t \cdot v_2 &= t^{-nq} v_2, \quad q \in \mathbb{N}, \quad v_2 \in V_2, \quad n \in \mathbb{N}, (p, q) = 1. \end{aligned}$$

Let us introduce the following notation.

$$\begin{aligned} Z &= V//\widetilde{T}; \\ Y &= \{ \underbrace{v_1 \otimes \dots \otimes v_1}_q \otimes \underbrace{v_2 \otimes \dots \otimes v_2}_p \} \subset V_1^{\otimes q} \otimes V_2^{\otimes p}. \end{aligned}$$

**Proposition 6.** *There exists an  $\mathrm{SL}_2$ -equivariant isomorphism between the varieties  $Y$  and  $Z$ .*

*Proof.* Let us fix the isomorphism of  $\mathrm{SL}_2$ -modules  $V_1 \cong V_2$ . Let us also fix the bases in  $V_1$  and  $V_2$  corresponding to each other under the chosen isomorphism. Let

$x_1, x_2$  be the coordinates in  $V_1$  corresponding to the chosen basis and  $y_1, y_2$  be the corresponding coordinates in  $V_2$ . Then

$$\mathbb{K}[Y] = \mathbb{K}[Z] = \mathbb{K}[x_1^m x_2^{q-m} y_1^l y_2^{p-l} \mid m = 0, \dots, q; l = 0, \dots, p].$$

Hence,  $Y \xrightarrow{\text{SL}(2)} Z$ .

□

In the sequel we shall not distinguish between  $Z$  and  $Y$  and use the notation  $Y$ . Let us consider the following  $\text{SL}_2$ -orbit in  $Y$

$$\mathcal{O} = \text{Orb}(u), \quad u = \underbrace{e_1 \otimes \dots \otimes e_1}_q \otimes \underbrace{e_2 \otimes \dots \otimes e_2}_p.$$

Here  $e_1 \in V_1$  and  $e_2 \in V_2$  are not proportional after the identification of  $V_1$  and  $V_2$  by the chosen isomorphism. The stabilizer of  $u$  is  $St_u = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \varepsilon^{|q-p|} = 1 \right\}$ . If  $p \neq q$ , then  $St_u \cong \mathbb{Z}_{q-p}$  is a finite group. Hence  $\text{Orb}(u)$  is three-dimensional and open. If  $p = q$ , then the stabilizer of any point contains a one-dimensional torus. Therefore there is no open orbit. In the sequel we shall consider the case  $q > p$ . Then the toric variety  $Y = V//\tilde{T}$  contains an open  $\text{SL}(2)$ -orbit. Therefore,  $Y$  is a toric  $\text{SL}(2)/\mathbb{Z}_{q-p}$ -embedding.

The group  $\text{SL}(2)$  acts on itself by left shifts. This action induces the action on the homogeneous space  $\text{SL}(2)/\mathbb{Z}_r$ . Let us consider the  $\text{SL}(2)$ -equivariant dominant morphism

$$\varphi: \text{SL}(2) \rightarrow Y; \quad g \mapsto g \cdot u.$$

It corresponds to the  $\text{SL}(2)$ -equivariant embedding

$$\varphi^*: \mathbb{K}[Y] \hookrightarrow \mathbb{K}[\text{SL}(2)] = \mathbb{K}[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1).$$

Suppose  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2)$ . Then  $g \cdot e_1 = ae_1 + ce_2$ ,  $g \cdot e_2 = be_1 + de_2$ . Note that  $g \cdot \alpha = d\alpha - b\gamma$ ,  $g \cdot \gamma = a\gamma - c\alpha$ ,  $g \cdot \beta = d\beta - b\delta$ ,  $g \cdot \delta = a\delta - c\beta$ . We have

$$\begin{aligned} \varphi^*(x_1^m x_2^{q-m} y_1^l y_2^{p-l})(g) &= x_1^m x_2^{q-m} y_1^l y_2^{p-l}(\varphi(g)) = \\ &= x_1^m x_2^{q-m} y_1^l y_2^{p-l}(\underbrace{g \cdot e_1 \otimes \dots \otimes g \cdot e_1}_q \otimes \underbrace{g \cdot e_2 \otimes \dots \otimes g \cdot e_2}_p) = \\ &= x_1^m x_2^{q-m} y_1^l y_2^{p-l}(\underbrace{(ae_1 + ce_2) \otimes \dots \otimes (ae_1 + ce_2)}_q \otimes \underbrace{(be_1 + de_2) \otimes \dots \otimes (be_1 + de_2)}_p) = \\ &= a^m c^{q-m} b^l d^{p-l} = \alpha^m \gamma^{q-m} \beta^l \delta^{p-l}(g). \end{aligned}$$

Thus  $\varphi^*(x_1^m x_2^{q-m} y_1^l y_2^{p-l}) = \alpha^m \gamma^{q-m} \beta^l \delta^{p-l}$ .

Let us consider a unipotent subgroup  $U = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$  in  $\text{SL}(2)$ . The algebra of  $U$ -invariant functions on  $\text{SL}(2)$  is  $\mathbb{K}[\text{SL}(2)]^U = \mathbb{K}[\alpha, \beta]$ . The restriction of  $\varphi^*$  to  $\mathbb{K}[Y]^U$  is  $\varphi^*|_{\mathbb{K}[Y]^U}: \mathbb{K}[Y]^U \hookrightarrow \mathbb{K}[\text{SL}(2)]^U = \mathbb{K}[\alpha, \beta]$ . Let us prove that the height of  $Y$  is equal to the maximal possible value of the fraction  $w/t$ , where  $\alpha^t \beta^w \in \text{Im } \varphi^*|_{\mathbb{K}[Y]^U} = \text{Im } \varphi^* \cap \mathbb{K}[\alpha, \beta]$ . Let  $W$  be the toric  $\text{SL}(2)$ -embedding such that  $Y = W//\mathbb{Z}_s$ . Then we have dominant morphisms

$$\text{SL}(2) \xrightarrow{\psi} W \xrightarrow{\pi} Y,$$

where  $\varphi = \pi \circ \psi$ . Hence,

$$\mathbb{K}[Y]^U \xrightarrow{\pi^*|_{\mathbb{K}[W]^U}} \mathbb{K}[W]^U \xrightarrow{\psi^*|_{\mathbb{K}[\mathrm{SL}(2)]^U}} \mathbb{K}[\alpha, \beta].$$

If  $\alpha^t \beta^w \in \mathrm{Im}(\varphi^*) \cap \mathbb{K}[\alpha, \beta]$ , then  $\alpha^t \beta^w \in \mathrm{Im}(\psi^*) \cap \mathbb{K}[\alpha, \beta]$ . Therefore, the height of  $W$ , which is equal to the height of  $Y$ , is not less than  $w/t$ . If the height of  $W$  is equal to  $h$ , then there exists a monomial  $\alpha^\xi \beta^\eta$  in  $\mathrm{Im}(\psi^*) \cap \mathbb{K}[\alpha, \beta]$  such that  $\eta/\xi = h$ . Then  $\alpha^\xi \beta^\eta = \psi^*(f)$  for some  $f \in \mathbb{K}[W]^U$ . Let  $\zeta = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$  be a generator of  $\mathbb{Z}_s$ . Suppose

$$\tilde{f} = f(\zeta \cdot f)(\zeta^2 \cdot f) \dots (\zeta^{s-1} \cdot f) \in \mathbb{K}[Y]^U.$$

It is easy to see that  $\varphi^*(\tilde{f}) = \alpha^{s\xi} \beta^{s\eta}$ . But  $\frac{s\eta}{s\xi} = \frac{\eta}{\xi} = h$ . Thus, the height of  $Y$  is the maximal possible value of  $w/t$ .

**Proposition 7.** *If a monomial  $\alpha^t \beta^w$  belongs to the image of  $\varphi^*$ , then  $w/t \leq p/q$ .*

*Proof.* Assume that  $\varphi^*(\sum z_i x_1^{m_i} x_2^{q-m_i} y_1^{l_i} y_2^{p-l_i}) = \alpha^t \beta^w, z_i \in \mathbb{K}$ . Then

$$\sum z_i \alpha^{m_i} \gamma^{q-m_i} \beta^{l_i} \delta^{p-l_i} = \alpha^t \beta^w + (\alpha\delta - \beta\gamma - 1)F(\alpha, \beta, \gamma, \delta).$$

(This equality holds in  $\mathbb{K}[\alpha, \beta, \gamma, \delta]$ .) Putting  $\gamma = 0$  we obtain

$$\sum z_i \alpha^q \beta^{l_i} \delta^{p-l_i} = \alpha^t \beta^w + (\alpha\delta - 1)\tilde{F}(\alpha, \beta, \gamma).$$

Substituting  $1/\alpha$  for  $\delta$  we obtain

$$\sum z_i \alpha^{q-p+l_i} \beta^{l_i} = \alpha^t \beta^w.$$

Therefore,

$$t = q - p + l_i; \quad w = l_i; \quad w/t = l_i/(q - p + l_i) \leq p/q.$$

□

For the monomial  $\alpha^q \beta^p = \varphi^*(x_1^q y_1^p)$ , we have  $w/t = p/q$ . Hence the height of  $Y$  is equal to  $p/q$ . But, by definition, the height of  $Y$  is equal to the height of  $W$ . And it is equal to the height of  $X$ . Therefore the height of  $X$  is equal to  $p/q$ . Recall that  $Y = V//\tilde{T}$  is an  $\mathrm{SL}(2)/\mathbb{Z}_{q-p}$ -embedding, and  $X = (V//\tilde{T})//\mathbb{Z}_l = Y//\mathbb{Z}_l$ . Hence the order of the stabilizer of a point in the open  $\mathrm{SL}(2)$ -orbit in  $X$  is divisible by  $q - p$ .

Let us check that an  $\mathrm{SL}(2)/\mathbb{Z}_{(q-p)l}$ -embedding of height  $p/q$  is toric for every positive integer  $l$ . Indeed,  $Y = \mathbb{K}^4//\tilde{T}$  is a toric variety with the height  $p/q$ , that is,  $Y$  corresponds to  $l = 1$ . Let us consider an  $\mathrm{SL}(2)/\mathbb{Z}_{(q-p)l}$ -embedding with height  $p/q$ . We have  $X = W//\mathbb{Z}_{(q-p)l} = (W//\mathbb{Z}_{(q-p)})//\mathbb{Z}_l = Y//\mathbb{Z}_l$ . Suppose  $\pi: Y \rightarrow X$  is the factorisation morphism. The torus  $T = \overline{T}/\tilde{T}$  acts on  $Y$  with an open orbit and trivial stabilizer of generic point. Let us define a  $T$ -action on  $X$  by  $t \cdot \pi(y) = \pi(t \cdot y)$ . This action is well defined if the actions of  $T$  and  $\mathbb{Z}_l$  on  $Y$  commute. Let us check that the actions of  $T$  and  $\mathbb{Z}_l = \mathbb{Z}_{(q-p)l}/\mathbb{Z}_{q-p}$  on  $\mathbb{K}[Y]$  commute, where  $\mathbb{Z}_{(q-p)} \subset \mathrm{SL}(2)$  and  $\mathbb{Z}_{(q-p)l} \subset \mathrm{SL}(2)$ . Suppose

$$t = \tilde{t}\tilde{T} \in T, \quad \tilde{t} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \in \overline{T},$$

$$\bar{\zeta} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \in \mathbb{Z}_{l(q-p)}, \quad \zeta = \bar{\zeta}_{\mathbb{Z}_{q-p}} \in \mathbb{Z}_l.$$

Here  $\varepsilon$  is a  $(q-p)l$ -th root of unity. Then

$$t \cdot x_1^m x_2^{q-m} y_1^n y_2^{p-n} = a^{-m} b^{m-q} c^{-n} d^{n-p} x_1^m x_2^{q-m} y_1^n y_2^{p-n}.$$

Recall that

$$Y = \{v_1 \otimes \dots \otimes v_1 \otimes v_2 \otimes \dots \otimes v_2\} \subset V_1^{\otimes q} \otimes V_2^{\otimes p}.$$

The open  $\mathrm{SL}(2)$ -orbit is  $\{g \cdot e_1 \otimes \dots \otimes g \cdot e_1 \otimes g \cdot e_2 \otimes \dots \otimes g \cdot e_2\}$ , where  $(e_1, e_2)$  is a fixed basis in  $\mathbb{K}^2$ . Suppose  $v = g_0 \cdot e_1 \otimes \dots \otimes g_0 \cdot e_1 \otimes g_0 \cdot e_2 \otimes \dots \otimes g_0 \cdot e_2$ . Then

$$\begin{aligned} \zeta \cdot v &= g_0 \bar{\zeta} \cdot e_1 \otimes \dots \otimes g_0 \bar{\zeta} \cdot e_1 \otimes g_0 \bar{\zeta} \cdot e_2 \otimes \dots \otimes g_0 \bar{\zeta} \cdot e_2 = \\ &= \varepsilon g_0 \cdot e_1 \otimes \dots \otimes \varepsilon g_0 \cdot e_1 \otimes \varepsilon^{-1} g_0 \cdot e_2 \otimes \dots \otimes \varepsilon^{-1} g_0 \cdot e_2 = \\ &= \varepsilon^{q-p} g_0 \cdot e_1 \otimes \dots \otimes g_0 \cdot e_1 \otimes g_0 \cdot e_2 \otimes \dots \otimes g_0 \cdot e_2 = \varepsilon^{q-p} v. \end{aligned}$$

Hence,  $\zeta \cdot x_1^m x_2^{q-m} y_1^n y_2^{p-n}(v) = x_1^m x_2^{q-m} y_1^n y_2^{p-n}(\zeta^{-1} \cdot v) = \varepsilon^{p-q} x_1^m x_2^{q-m} y_1^n y_2^{p-n}(v)$ . This implies that the actions of the torus and  $\mathbb{Z}_l$  commute. Thus,  $X$  is a toric  $\mathrm{SL}(2)/\mathbb{Z}_{(q-p)l}$ -embedding.

Let us formulate the main result.

**Theorem 2.** *Let  $X$  be an irreducible three-dimensional normal affine variety. Assume that there exists a locally transitive  $\mathrm{SL}(2)$ -action on  $X$ . Then  $X$  is toric if and only if  $X$  is an  $\mathrm{SL}(2)/\mathbb{Z}_r$ -embedding of height  $\frac{p}{q}$ , where  $(p, q) = 1$  and  $r$  is divisible by  $q - p$ .*

**Corollary 3.** *An  $\mathrm{SL}(2)$ -embedding is toric if and only if its height can be written as  $p/(p+1)$ .*

## 7. THE CONE OF A TORIC $\mathrm{SL}(2)$ -EMBEDDING

As it was mentioned in Section 1 any toric variety  $X$  corresponds to a polyhedral cone in the space  $N_{\mathbb{Q}} \cong \mathbb{Q}^{\dim X}$ . Let us describe the cone corresponding to an  $\mathrm{SL}(2)/\mathbb{Z}_{(q-p)l}$ -embedding  $X$  of height  $p/q$ .

In the previous section we gave an explicit construction of  $X$  as the quotient of a toric  $\mathrm{SL}(2)/\mathbb{Z}_{(q-p)}$ -embedding  $Y$  by the action of the group  $\mathbb{Z}_l$ . The algebra of regular functions on  $Y$  is  $\mathbb{K}[x_1^m x_2^{q-m} y_1^n y_2^{p-n}]$ ,  $0 \leq m \leq q$ ,  $0 \leq n \leq p$ . Moreover, we obtained an explicit formula for the action of  $\mathbb{Z}_l$  on  $\mathbb{K}[Y]$ . The action of the generator  $\zeta$  of  $\mathbb{Z}_l$  on  $\mathbb{K}[Y]$  is given by  $\zeta \cdot x_1^m x_2^{q-m} y_1^n y_2^{p-n} = \varepsilon^{p-q} x_1^m x_2^{q-m} y_1^n y_2^{p-n}$ , where  $\varepsilon$  is an  $l(q-p)$ -th root of unity. Hence the algebra of invariants is

$$\mathbb{K}[X] = \mathbb{K}[Y]^{\mathbb{Z}_l} = \mathbb{K}[x_1^u x_2^{lq-u} y_1^v y_2^{lp-v}].$$

Let us embed  $\mathbb{K}[X]$  in the algebra of polynomials in three variables. Suppose  $f = x_2/x_1, g = y_2/y_1, h = x_1^{lq} y_1^{lp}$  in  $\mathbb{K}(X)$ . It is clear that  $f, g$  and  $h$  are algebraically independent. Then  $x_1^{lq-u} x_2^u y_1^{lp-v} y_2^v = f^u g^v h$ . There is a natural embedding  $\mathbb{K}[f, g, h] \hookrightarrow \mathbb{K}[f, g, h, f^{-1}, g^{-1}, h^{-1}]$ . It corresponds to the embedding of the torus  $T = \mathrm{Spec} \mathbb{K}[f, g, h, f^{-1}, g^{-1}, h^{-1}]$  into the three-dimensional affine space  $V = \mathrm{Spec} \mathbb{K}[f, g, h]$ . Recall that  $M$  is the lattice of characters of  $T$ . Then  $M$  is spanned by the vectors  $a = (1, 0, 0), b = (0, 1, 0)$  and  $c = (0, 0, 1)$ . Let  $N$  be the dual lattice. Then  $\mathbb{K}[f, g, h]$  is the semigroup algebra of the semigroup  $P$  generated by  $a, b$  and  $c$ . Let us define an isomorphism  $i: \mathbb{K}[f, g, h] \rightarrow \mathbb{K}[P]$  by  $i(f) = a, i(g) = b, i(h) = c$ . Then  $i(f^u g^v h) = ua + vb + c$ .

Denote by  $\sigma$  the cone corresponding to  $X$  and by  $\hat{\sigma}$  the cone dual to  $\sigma$ . The cone  $\hat{\sigma}$  is spanned by the vectors

$$ua + vb + c, u \in \mathbb{Z} \cap [0, lq], v \in \mathbb{Z} \cap [0, lp].$$

Then the cone  $\sigma$  consists of all vectors  $w \in N$  such that  $\langle w, ua + vb + c \rangle \geq 0$ . A linear function accepts the minimal value at an end of an interval. Hence the cone  $\sigma$  can be given by the following inequalities:

$$(w, c) \geq 0, (w, lqa + c) \geq 0, (w, lpb + c) \geq 0, (w, lqa + lpb + c) \geq 0.$$

To find edges of the cone  $\sigma$  we should choose two of the above four inequations, replace them by the corresponding equations, and solve the obtained system of equations. Besides, we need to choose only those solutions that belong to  $\sigma$ . We have six systems.

1)

$$\begin{cases} t = 0; \\ rlq + t = 0. \end{cases} \quad \text{Hence } r = 0, t = 0. \text{ The answer is } (0, 1, 0).$$

2)

$$\begin{cases} t = 0; \\ slp + t = 0. \end{cases} \quad \text{Therefore } s = 0, t = 0. \text{ The answer is } (1, 0, 0).$$

3)

$$\begin{cases} t = 0; \\ rlq + slp + t = 0. \end{cases} \quad \begin{array}{l} \text{Hence } t = 0, rlq + slp = 0. \text{ Since } rlq \geq 0 \text{ and } slp \geq 0, \\ \text{we obtain } rlq = slp = 0, \text{ that is, } r = s = t = 0. \end{array}$$

4)

$$\begin{cases} rlq + t = 0; \\ slp + t = 0. \end{cases} \quad \begin{array}{l} \text{Therefore } 0 \leq rlq + slp + t = -t \leq 0. \text{ Hence } r = s = \\ t = 0. \end{array}$$

5)

$$\begin{cases} rlq + t = 0; \\ rlq + slp + t = 0. \end{cases} \quad \begin{array}{l} \text{Therefore } s = 0. \text{ Since for vector } (1, 0, -lq) \text{ we ob-} \\ \text{tain } slp + t < 0, \text{ the answer is } (-1, 0, lq). \end{array}$$

6)

$$\begin{cases} slp + t = 0; \\ rlq + slp + t = 0. \end{cases} \quad \text{Hence } r = 0. \text{ The answer is } (0, -1, lp).$$

We obtain four 1-dimensional faces:  $\rho_1 = \mathbb{Q}_+(1, 0, 0)$ ,  $\rho_2 = \mathbb{Q}_+(0, 1, 0)$ ,  $\rho_3 = \mathbb{Q}_+(-1, 0, lq)$  and  $\rho_4 = \mathbb{Q}_+(0, -1, lp)$ . Let us formulate the result.

**Proposition 8.** *The cone corresponding to an  $SL(2)/\mathbb{Z}_{l(q-p)}$ -embedding of height  $p/q$  regarded as a toric variety is*

$$\text{cone}((1, 0, 0), (0, 1, 0), (-1, 0, lq), (0, -1, lp)).$$

## 8. FINAL REMARKS

Recently the question on degeneracy of an algebraic variety to a toric one has been actively studied (see, for example, [11]). In the case of an affine  $SL(2)/\mathbb{Z}_r$ -embedding the standard procedure of contraction of an action [12] gives us a toric variety. But if the starting variety was toric, then the variety after contraction is never isomorphic to it.

Note that the result of this paper can be consider as the first step towards describing the group of all (not equivariant) automorphisms of an  $SL(2)/\mathbb{Z}_r$ -embedding. Indeed, the rank of this group has been calculated. For toric  $SL(2)/\mathbb{Z}_r$ -embeddings it is equal to 3. For non-toric  $SL(2)/\mathbb{Z}_r$ -embeddings it equals 2, because a 2-dimensional torus acts on any  $SL(2)/\mathbb{Z}_r$ -embedding (see [5, chapter 3, p. 4.8]).

## REFERENCES

- [1] D. Fulton, *Introduction to toric varieties* // Annals of Math. Studies **131**, Princeton University Press, Princeton, NJ, 1993.
- [2] D.A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Alg. Geometry **4** (1995), 17-50.
- [3] H. Kraft, V.L. Popov, *Semisimple group actions on the three dimensional affine space are linear*, Comment. Math. Helvetici **60** (1985), 466-479.
- [4] V.L. Popov, *Quasihomogeneous affine algebraic varieties of the group  $SL(2)$*  (Russian), Izvestiya AN SSSR, Ser. Mat. **37:4** (1973), 792-832; English transl. in Math. USSR-Izvestiya **7:4** (1973), 793-831.
- [5] H. Kraft, *Geometrische Methoden in der Invariantentheorie* (German) // Friedr. Vieweg&Sohn, Braunschweig, 1984.
- [6] D.I. Panyushev, *The canonical module of a quasihomogeneous normal affine  $SL_2$ -variety*, Mat. Sbornik **182:8** (1991), 1211-1221; English transl. in Math. USSR-Sbornik **73:2** (1992), 569-578.
- [7] F. Berchtold, J. Hausen, *Homogeneous coordinates for algebraic varieties*, J. Algebra **266** (2003), 636-670.
- [8] Y. Hu, S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331-348.
- [9] F. Knop, H. Kraft, T. Vust, *The Picard group of a  $G$ -variety*, in: Algebraische Transformationsgruppen und Invariantentheorie, DMV Seminar, Vol. **13**, Birkhäuser, Basel (1989), 77-87.
- [10] V.L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles* (Russian), Izvestiya AN SSSR, Ser. Mat. **38:2** (1974), 294-332; English transl. in Math. USSR-Izvestiya **8:2** (1974), 301-327.
- [11] V. Alexeev, M. Brion, *Toric degenerations of spherical varieties*, Selecta Math. **10** (2004), 453-478.
- [12] V.L. Popov, *Contractions of the actions of reductive algebraic groups*, Mat. Sbornik **130:3** (1986), 310-334; English transl. in Math. USSR-Sbornik **58:2** (1987), 311-335.

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